State space models for multi-setup operational modal analysis

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ABSTRACT: Operational Modal Analysis consists on estimate the modal parameters of a structure (natural frequencies, damping ratios and modal vectors) from output-only vibration measurements. The modal vectors can be only estimated where a sensor is placed, so when the number of available sensors is lower than the number of tested points, it is usual to perform several tests changing the position of the sensors from one test to the following (multiple setups of sensors): some sensors stay at the same position from setup to setup, and the other sensors change the position until all the tested points are covered. The permanent sensors are then used to merge the mode shape estimated at each setup (or partial modal vectors) into global modal vectors.

Traditionally, the partial modal vectors are estimated independently setup by setup, and the global modal vectors are obtained in a postprocess phase. In this work we present two state space models that can be used to process all the recorded setups at the same time, and we also present how these models can be estimated using the maximum likelihood method. The result is that the global mode shape of each mode is obtained automatically, and subsequently, a single value for the natural frequency and damping ratio of the mode is computed. Finally, both models are compared using real measured data.

KEY WORDS: Operational Modal Analysis, multiple setups of sensors, state space model, Expectation-Maximization algorithm

1 INTRODUCTION

The estimation of modal parameters from experimental data is a well-established discipline in mechanical engineering [1], [2]. The procedure consists on to apply a known dynamic force to the structure and to record the response of the structure due to this force. After that, force and response are used to estimate the modal parameters (Experimental Modal Analysis or EMA is the name used for this technique in technical literature).

The application of this method to large structures like bridges or buildings has some drawbacks: first, because it is difficult and often expensive to apply a measured force to these structures; and second, because apart from the measured inputs, the structure is also excited by other unmeasured forces like wind, traffic, earthquakes,...The option is to estimate the modal parameters using only the response of the structure. This response is usually due to ambient loads (for example, wind), or to operational loads (traffic, human loading). That's why this procedure is called Operational Modal Analysis (or OMA), ambient modal analysis or output-only modal analysis.

The parametric approach to estimate the modal parameters consists on to fit a mathematical model to the vibration data, and then to compute the modal parameters from the model parameters. In EMA, both input and output data are used to estimate the model. In OMA, where the input are unknown, it is assumed that they are realizations of a stochastic process with known properties (generally, white noise processes with zero mean and a given variance). The most successful approach in the time domain is to use the well-known state space model:

$$x_t = Ax_{t-1} + w_t \tag{1a}$$
$$y_t = Cx_t + v_t, \tag{1b}$$

where $y_t \in \mathbb{R}^{n_o}$ is the vibration measured at n_o different degrees of freedom (DOFs); $A \in \mathbb{R}^{n_s \times n_s}$ and $C \in \mathbb{R}^{n_o \times n_s}$ are model parameters, and $w_t \in \mathbb{R}^{n_s}$, $v_t \in \mathbb{R}^{n_o}$ are white noise processes with zero mean and covariance matrices $Q \in \mathbb{R}^{n_s \times n_s}$ and $R \in \mathbb{R}^{n_o \times n_o}$ respectively.

If the state space model is estimated from vibration data recorded at a structural system, the modal parameters of the structure can be computed from the parameters of the state space model, in particular from matrices A and C (see [3]). It is usual to express the jth eigenvalue of A as

$$\lambda_j = \exp\left[\left(-\zeta_j\omega_j + i\omega_j\sqrt{1-\zeta_j^2}\right)\Delta t\right], \tag{2}$$

where Δt is the time step. Therefore

$$\omega_j = \frac{\left|\ln(\lambda_j)\right|}{\Delta t}, \tag{3}$$

$$\zeta_j = \frac{-\text{Real}\left[\ln(\lambda_j)\right]}{\omega_j \Delta t}. \tag{4}$$

If proportional damping is admitted, ω_j is the undamped natural frequency and ζ_j is the damping ratio.

The jth mode shape $\psi_j \in \mathbb{C}^{n_o}$ evaluated at sensor locations can be obtained using the following expression:

$$\psi_j = Cv_j, \tag{5}$$

where v_j is the eigenvector corresponding to the eigenvalue λ_j. In general, the modal vectors are complex. If proportional damping is admitted, the modal vectors are real; we can take the real part of ψ_j:

$$\phi_j = \text{Real}[Cv_j] \in \mathbb{R}^{n_o}. \tag{6}$$

It is important to note that the number of modes estimated from the data y_t depends on the dimension of matrix A, that is, n_s. Since each pair of complex and complex-conjugate eigenvalues gives a mode of vibration, the number of modes estimated is $n_s/2$ (assuming all the eigenvalues are complex). The value n_s is not known in advance, it has to be estimated using specific techniques (see [4]) .

Another important aspect is that, taking into account Equation (5), the size of the estimated modal vectors is equal to the number of measured outputs, n_o. In fact, the modal vectors are estimated only at the measured DOFs. This is important because often the number of required DOFs is higher than the number of available sensors (either because the structure is large or because the resolution for the modal vectors is high). The solution adopted in technical literature consists in to measure the vibration of the structure changing the positions of the sensors (each sensor position is called *setup of sensors*): some sensors do not move from setup to setup (the permanent sensors), but the rest of available sensors change their position from setup to setup (moving sensors), so that all the required DOFs are measured. Then, the information recorded in the different setups has to be processed properly to obtain the modal vectors in all the measured points.

The traditional approach consisted in to estimate the partial modal vectors (those corresponding to the DOFs measured in each setup), and then, in a post-process phase, to build the global modal vectors (corresponding to all the measured DOFs) taking into account the information given by the permanent sensors. This process is usually tiresome, time consuming, and difficult.

On the other hand, there is an increasing interest in to process all the setups at the same time, so the global modal vectors are obtained directly. In the time domain, this approach requires to use the appropriate state space model and to develop the identification algorithms to estimate such a model.

In this work we analyse two different state space models that can be used to estimate the modal parameters from the data measured in tests with multiple setups of sensors. We use the maximum likelihood method and the EM algorithm to estimate both state space models. The models and the estimation algorithm are tested using data recorded at a steel frame structure with antennae attached at the top.

2 STATE SPACE MODELS FOR THE JOINT ESTIMATION OF MODAL PARAMETERS

In this section we show two state space models for the computation of the modal parameters. The first one, called here *state space model 1*, was first used in [5]. More datails on the second model, *state space model 2*, can be found in [6].

2.1 State space model 1

The data recorded by n_o sensors simultaneously placed in a structure can be represented by:

$$y_t = \begin{bmatrix} d_{1,t} \\ d_{2,t} \\ \cdots \\ d_{n_o,t} \end{bmatrix}, \tag{7}$$

where $d_{j,t}$ stands for the datum measured by sensor j in the time instant t. The data recorded by the n_o sensors in M different setups of sensors can be represented by:

$$y_t^{(1)} = \begin{bmatrix} d_{1,t}^{(1)} \\ d_{2,t}^{(1)} \\ \cdots \\ d_{n_o,t}^{(1)} \end{bmatrix}, y_t^{(2)} = \begin{bmatrix} d_{1,t}^{(2)} \\ d_{2,t}^{(2)} \\ \cdots \\ d_{n_o,t}^{(2)} \end{bmatrix}, \ldots, y_t^{(M)} = \begin{bmatrix} d_{1,t}^{(M)} \\ d_{2,t}^{(M)} \\ \cdots \\ d_{n_o,t}^{(M)} \end{bmatrix}, \tag{8}$$

where $d_{j,k}^{(s)}$ stands for the datum measured by sensor j at the time instant t and setup s (for simplicity, it is assumed that all the setups have the same number of sensors). These data can be also represented by:

$$y_t^{(1)} = \begin{bmatrix} y_{1,t}^{(1)} \\ y_{2,t}^{(1)} \end{bmatrix}, y_t^{(2)} = \begin{bmatrix} y_{1,t}^{(2)} \\ y_{2,t}^{(2)} \end{bmatrix}, \ldots, y_t^{(M)} = \begin{bmatrix} y_{1,t}^{(M)} \\ y_{2,t}^{(M)} \end{bmatrix}; \tag{9}$$

$y_{1,t}^{(s)} \in \mathbb{R}^{n_{o1}\times 1}$ stands for the data measured by the permanent sensors (consider there are n_{o1} permanent sensors) in setup s,

$$y_{1,t}^{(s)} = \begin{bmatrix} d_{1,t}^{(s)} \\ d_{2,t}^{(s)} \\ \cdots \\ d_{n_{o1},t}^{(s)} \end{bmatrix}; \tag{10}$$

and $y_{2,t}^{(s)} \in \mathbb{R}^{n_{o2}\times 1}$ stands for the data measured by the moving sensors (there are $n_{o2} = n_o - n_{o1}$ moving sensors) in setup s.

$$y_{2,t}^{(s)} = \begin{bmatrix} d_{n_{o1}+1,t}^{(s)} \\ d_{n_{o1}+2,t}^{(s)} \\ \cdots \\ d_{n_o,t}^{(s)} \end{bmatrix}; \tag{11}$$

The state space model we can use with the data of the M setups (Equation (9)) is

$$x_{t+1}^{(r)} = Ax_t^{(r)} + w_t^{(r)} \tag{12a}$$

$$\begin{bmatrix} y_{1,t}^{(r)} \\ y_{2,t}^{(r)} \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2^{(r)} \end{bmatrix} x_t^{(r)} + \begin{bmatrix} v_{1,t}^{(r)} \\ v_{2,t}^{(r)} \end{bmatrix}, \tag{12b}$$

$$r = 1, 2, \ldots, M. \tag{12c}$$

Note the observation equation can be also written as

$$y_t^{(r)} = C^{(r)}x_t^{(r)} + v_t^{(r)}, \tag{13}$$

where

$$y_t^{(r)} = \begin{bmatrix} y_{1,t}^{(r)} \\ y_{2,t}^{(r)} \end{bmatrix}; \quad C^{(r)} = \begin{bmatrix} C_1 \\ C_2^{(r)} \end{bmatrix}; \quad v_t^{(r)} = \begin{bmatrix} v_{1,t}^{(r)} \\ v_{2,t}^{(r)} \end{bmatrix}. \tag{14}$$

For the noise processes we assume Gaussian white noise process

$$w_t^{(r)} \rightsquigarrow N(0, Q^{(r)}), \quad v_t^{(r)} \rightsquigarrow N(0, R^{(r)}). \tag{15}$$

The features of this model are:

• The measurements of the permanent sensors must be placed at the first n_{o1} rows of the output vector in all setups, $y_t^{(r)}$, $r = 1,2,\dots,M$;
• The matrix A is the same for all the setups: this matrix represents the dynamics of the structure, and it is assumed the structure is time invariant, it does not change from setup to setup;
• The states are different for all the setups because the measured data are different;
• The output matrices $C^{(r)}$ have two parts: the n_{o1} first rows (corresponding to the permanent sensors) are constant for all the setups, and the rest rows are record-depending;
• The global C matrix (the matrix corresponding to all the measured DOFs) can be constructed as:

$$C = \begin{bmatrix} C_1 \\ C_2^{(1)} \\ C_2^{(2)} \\ \dots \\ C_2^{(M)} \end{bmatrix} \in \mathbb{R}^{(n_{o1}+n_{o2}M)\times n_s} \tag{16}$$

• The noise process corresponding to the states, w_t, is record-depending. In OMA the unobserved inputs are modelled by this noise process, so the model takes into account the possibility of inputs with different variance;
• The noise process for the outputs, v_t, is also different for each setup.
The unknown parameters of this model are

$$\theta = \{A, C^{(r)}, Q^{(r)}, R^{(r)}, \bar{x}_0^{(r)}, P_0^{(r)}\}, \quad r = 1,2,\dots,M, \tag{17}$$

where $\bar{x}_0^{(r)}$ and $P_0^{(r)}$ are the mean and variance of the initial state $x_0^{(r)}$ respectively (which is assumed to be normal distributed).

2.2 State space model 2

Imagine you have four accelerometers and you need to measure the response of one structure at eight different DOFs, so you decide to use two accelerometers as permanent sensors and the other two as moving sensors. The resulting three setups of sensors are given in Table 1.

setup	measured DOFs			
1	1	2	3	4
2	1	2	5	6
3	1	2	7	8

Table 1. Example: measured DOFs per setup.

The data recorded by the four accelerometers at time instant t in each setup can be expressed as

$$y_t^{(1)} = \begin{bmatrix} \ddot{q}_{1,t}^{(1)} \\ \ddot{q}_{2,t}^{(1)} \\ \ddot{q}_{3,t}^{(1)} \\ \ddot{q}_{4,t}^{(1)} \end{bmatrix}, \quad y_t^{(2)} = \begin{bmatrix} \ddot{q}_{1,t}^{(2)} \\ \ddot{q}_{2,t}^{(2)} \\ \ddot{q}_{5,t}^{(2)} \\ \ddot{q}_{6,t}^{(2)} \end{bmatrix}, \quad y_t^{(3)} = \begin{bmatrix} \ddot{q}_{1,t}^{(3)} \\ \ddot{q}_{2,t}^{(3)} \\ \ddot{q}_{7,t}^{(3)} \\ \ddot{q}_{8,t}^{(3)} \end{bmatrix}, \tag{18}$$

where $\ddot{q}_{j,t}^{(s)}$ represents the acceleration measured in DOF j at time instant t and setup s.

Apart from model (12), we can use the following state space model for these data

$$x_{t+1}^{(r)} = A x_t^{(r)} + w_t^{(r)} \tag{19a}$$
$$y_t^{(r)} = L^{(r)} C x_t^{(r)} + v_t^{(r)}, \tag{19b}$$
$$r = 1,2,\dots,M, \tag{19c}$$

($M = 3$ in this example). $L^{(r)} \in \mathbb{R}^{n_o \times n_{og}}$ is a location matrix formed by ones and zeros (n_{og} is the number of total DOFs measured in all the setups, $n_{og} = n_{o1} + M \cdot n_{o2}$): $L_{jk}^{(r)} = 1$ if the jth sensor in the rth setup measures the kth DOF of the global list of measured DOFs; and zero otherwise. The location matrices are known for each setup r.

In order to clarify the definition of location matrices, we show next the location matrices for the three setups given in Table 1. First, the global list of measured DOFs is:

$$y_t^{(g)} = \begin{bmatrix} \ddot{q}_{1,t} \\ \ddot{q}_{2,t} \\ \ddot{q}_{3,t} \\ \ddot{q}_{4,t} \\ \ddot{q}_{5,t} \\ \ddot{q}_{6,t} \\ \ddot{q}_{7,t} \\ \ddot{q}_{8,t} \end{bmatrix} \in \mathbb{R}^{n_{og}\times 1} \tag{20}$$

($n_{og} = n_{o1} + M \cdot n_{o2} = 2 + 3 \cdot 2 = 8$). The following relationships must be verified

$$y_t^{(1)} = L^{(1)} y_t^{(g)}, \quad y_t^{(2)} = L^{(2)} y_t^{(g)}, \quad y_t^{(3)} = L^{(3)} y_t^{(g)}; \tag{21}$$

Thus

$$L^{(1)} = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \end{bmatrix};$$

$$L^{(2)} = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \end{bmatrix};$$

$$L^{(3)} = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \end{bmatrix}.$$

Note again the observation equation can be also written as

$$y_t^{(r)} = C^{(r)} x_t^{(r)} + v_t^{(r)}. \tag{22}$$

For the noise processes we assume Gaussian white noise process

$$w_t^{(r)} \rightsquigarrow N(0, Q^{(r)}), \quad v_t^{(r)} \rightsquigarrow N(0, R^{(r)}). \tag{23}$$

The features of this model are:
• The matrix A is the same for all the setups: this matrix represents the dynamics of the structure, and it is assumed the structure is time invariant, it does not change from setup to setup;

• The states are different for all the setups because the measured data are different;
• The measurement vectors, $y_t^{(r)},\ r=1,2,\ldots,M$, can be sorted in any order, mixing permanent and moving measurements.
• The global C matrix is obtained directly;
• The noise process corresponding to the states, w_t, is record-depending;
• The noise process for the outputs, v_t, is also different for each setup;
• The moving sensors can measure the same DOF in more than one setup, and the data recorded in those setups will be used to estimate the modal vectors at that DOF.

The unknown parameters of this model are

$$\theta = \{A, C, Q^{(r)}, R^{(r)}, \bar{x}_0^{(r)}, P_0^{(r)}\}, \quad r = 1,2,\ldots,M, \tag{24}$$

where $\bar{x}_0^{(r)}$ and $P_0^{(r)}$ are the mean and variance of the initial state $x_0^{(r)}$ respectively (which is assumed to be normal distributed).

2.3 Model estimation

To estimate the parameters of the state space models 1 and 2, we propose to use maximum likelihood estimation (MLE) with a maximization procedure based on the Expectation Maximization algorithm. The EM algorithm is a general-purpose method for MLE [7], that Shumway and Stoffer [8] used to estimate state space models. The performance of the EM algorithm for OMA was analysed in [9]. The algorithm is simple to apply since at each iteration the optimal solution for the unknown parameters can be obtained from explicit formulas.

Let be the observed outputs $Y_N^{(r)} = \{y_1^{(r)}, y_2^{(r)}, \ldots, y_N^{(r)}\}$ and the states $X_N^{(r)} = \{x_1^{(r)}, x_2^{(r)}, \ldots, x_N^{(r)}\}$. The probability density function for one individual record is given by (see [9])

$$\begin{aligned} & f_{\theta^{(r)}}(X_N^{(r)}, Y_N^{(r)}) = \\ & = f_{\bar{x}_0^{(r)}, P_0^{(r)}}(x_0^{(r)}) \prod_{t=1}^{N} f_{A,Q^{(r)}}(X_t^{(r)}|X_{t-1}^{(r)}) \prod_{t=1}^{N} f_{C,R^{(r)}}(Y_t^{(r)}|X_t^{(r)}). \end{aligned} \tag{25}$$

where under Gaussian assumption

$$\begin{aligned} f_{\bar{x}_0^{(r)}, P_0^{(r)}}(x_0^{(r)}) &= \frac{1}{(2\pi)^{n_s/2}|P_0^{(r)}|^{1/2}} \cdot \\ & \cdot \exp\left(-\frac{1}{2}(x_0^{(r)} - \bar{x}_0^{(r)})^T \left(P_0^{(r)}\right)^{-1} (x_0^{(r)} - \bar{x}_0^{(r)})\right), \end{aligned}$$

$$\begin{aligned} f_{A,Q^{(r)}}(X_t^{(r)}|X_{t-1}^{(r)}) &= \frac{1}{(2\pi)^{n_s/2}|Q^{(r)}|^{1/2}} \cdot \\ & \cdot \exp\left(-\frac{1}{2}(x_t^{(r)} - Ax_{t-1}^{(r)})^T \left(Q^{(r)}\right)^{-1} (x_t^{(r)} - Ax_{t-1}^{(r)})\right), \end{aligned}$$

$$\begin{aligned} f_{C,R^{(r)}}(Y_t^{(r)}|X_t^{(r)}) &= \frac{1}{(2\pi)^{n_o/2}|R^{(r)}|^{1/2}} \cdot \\ & \cdot \exp\left(-\frac{1}{2}(y_t^{(r)} - C^{(r)}x_t^{(r)})^T \left(R^{(r)}\right)^{-1} (y_t^{(r)} - C^{(r)}x_t^{(r)})\right). \end{aligned}$$

Thus, if we consider M independent setups, the joint density function $f_\theta(X_N, Y_N)$ will be the product of individual ones

$$f_\theta(X_N, Y_N) = \prod_{r=1}^{M} f_{\theta^{(r)}}(X_N^{(r)}, Y_N^{(r)}). \tag{26}$$

The complete data likelihood is defined by $L_{X_N,Y_N}(\theta) = f_\theta(X_N, Y_N)$. In practice the log-likelihood is used, so information is combined by addition and it can be written as a sum of the log-likelihood of each individual record:

$$l_{X_N,Y_N}(\theta) = \log L_{X_N,Y_N}(\theta) = \sum_{r=1}^{M} l_{X_N^{(r)}, Y_N^{(r)}}(\theta^{(r)}). \tag{27}$$

The log-likelihood of record r can be written as the sum of three uncoupled functions

$$l_{X_N^{(r)}, Y_N^{(r)}}(\theta^{(r)}) = -\frac{1}{2}[l_1(\mu_0^{(r)}, \Sigma_0^{(r)}) + l_2(A, Q^{(r)}) + l_3(C, R^{(r)}))], \tag{28}$$

where, ignoring constants, are

$$\begin{aligned} l_1(\bar{x}_0^{(r)}, P_0^{(r)}) =& \log|P_0^{(r)}| + \\ & + (x_0^{(r)} - \bar{x}_0^{(r)})^T \left(P_0^{(r)}\right)^{-1} (x_0^{(r)} - \bar{x}_0^{(r)}), \end{aligned}$$

$$\begin{aligned} l_2(A, Q^{(r)}) =& N\log|Q^{(r)}| + \\ & + \sum_{t=1}^{N} (x_t^{(r)} - Ax_{t-1}^{(r)})^T \left(Q^{(r)}\right)^{-1} (x_t^{(r)} - Ax_{t-1}^{(r)}), \end{aligned}$$

$$\begin{aligned} l_3(C, R^{(r)}) =& N\log|R^{(r)}| + \\ & + \sum_{t=1}^{N} (y_t^{(r)} - C^{(r)}x_t^{(r)})^T \left(R^{(r)}\right)^{-1} (y_t^{(r)} - C^{(r)}x_t^{(r)}). \end{aligned}$$

If we did have the observed vectors $Y_N^{(r)}$ and the states $X_N^{(r)}$, we could easily obtain the MLEs of the parameters θ (for example, using the results from multivariate normal theory). But the states are unknown (in fact, the states are unobserved quantities). The EM algorithm gives us an iterative method for finding the MLEs of θ using only the observed vectors $Y_N^{(r)}$, by successively maximizing the conditional expectation of the complete data likelihood (27). Two steps must be repeated iteratively:
• Expectation step. To compute the expected value of the log-likelihood (27), $\mathrm{E}[\mathrm{l}_{\mathrm{X_N,Y_N}}(\theta)|\mathrm{Y_N}, \theta_\mathrm{j}]$.
• Maximization step. To maximize $\mathrm{E}[\mathrm{l}_{\mathrm{X_N,Y_N}}(\theta)|\mathrm{Y_N}, \theta_\mathrm{j}]$ with respect to the parameters θ.

3 APPLICATIONS: STEEL-TRANSMITTER MAST

The results we show here correspond to a steel frame structure with antennae attached at the top (Figure 1). This structure has been deeply analysed in [4], [10] and [11].

On March 26, 1998, the structure was subject to ambient vibration measurements. The aim of the test was to investigate the structure's modal damping in the frequency range 0-5 Hz. Seventeen degrees of freedom, all horizontal accelerations, have been measured in three setups using three reference degrees of

Figure 1. Steel transmitter mast.

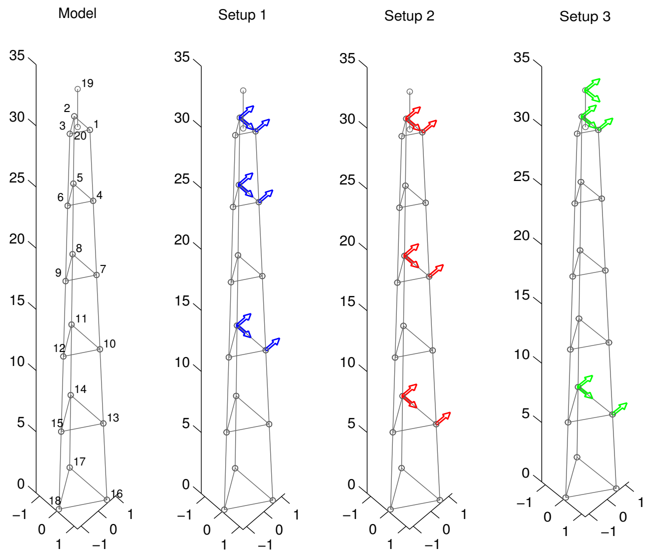

Figure 2. Steel mast grid and description of the three setups of sensors (the arrows indicate sensor position and measured direction).

freedom, that were common to each setup (Figure 2). Three horizontal accelerations have been measured at a height of 6.17, 12.17, 18.17, 24.17 and 29.90 m. The two orthogonal accelerations at the top of the mast (at height of 33.00 m) have been measured as well.

The data were sampled at a rate of 100 Hz. The cut-off frequency of the anti-aliasing filter that was used was set to 20 Hz. The number of samples was set to $N = 30720$, which resulted in a measurement time of approximately 5 min. Afterwards, the data were digitally filtered with a low-pass filter with a cut-off frequency of 5 Hz and resampled at 12.5 Hz, which reduced the number of samples to $N = 3840$.

State space model 1			
f	1.171 Hz	1.175 Hz	1.957 Hz
ζ	0,455%	0,881%	0,701%
1x	1.000	1.000	1.000
2x	1.112 + 0.167i	1.038 - 0.044i	-1.078 + 0.024i
2y	-1.279 - 2.870i	-0.259 + 0.307i	0.601 - 0.000i
4x	0.659 + 0.183i	0.634 + 0.064i	0.732 + 0.065i
5x	0.705 + 0.257i	0.663 + 0.052i	-0.856 - 0.036i
5y	-0.591 - 1.940i	-0.179 + 0.178i	0.482 + 0.044i
7x	0.362 + 0.100i	0.346 + 0.037i	0.480 + 0.035i
8x	0.382 + 0.184i	0.367 + 0.026i	-0.552 - 0.023i
8y	-0.321 - 1.118i	-0.083 + 0.105i	0.305 + 0.020i
10x	0.173 + 0.020i	0.163 - 0.005i	0.313 - 0.001i
11x	0.171 + 0.027i	0.162 - 0.001i	-0.333 + 0.0127i
11y	-0.189 - 0.462i	-0.035 + 0.046i	0.1980 + 0.001i
$13x^{(2)}$	0.050 + 0.006i	0.044 - 0.001i	0.148 - 0.003i
$13x^{(3)}$	0.072 + 0.014i	0.045 + 0.010i	0.145 - 0.003i
$14x^{(2)}$	0.047 + 0.012i	0.045 - 0.001i	-0.154 + 0.001i
$14x^{(3)}$	0.024 + 0.001i	0.046 - 0.014i	-0.154 + 0.001i
$14y^{(2)}$	-0.053 - 0.127i	-0.009 + 0.013i	0.090 + 0.000i
$14y^{(3)}$	-0.058 - 0.132i	-0.008 + 0.009i	0.089 + 0.001i
19x	2.088 + 0.658i	1.402 + 0.517i	-0.064 + 0.009i
19y	-1.883 - 4.790i	-0.399 + 0.122i	0.271 + 0.012i

Table 2. First three modes (natural frequencies, damping ratios and modal vectors) estimated using the state space model 1 and the EM algorithm. DOFs 13, 14 and 15 has been estimated in setup 2 and 3.

State space model 2			
f	1.171 Hz	1.179 Hz	1.958 Hz
ζ	0.290%	0.697%	0.610%
1x	1.000	1.000	1.000
2x	0.947 + 0.151i	1.045 - 0.040i	-1.039 + 0.004i
2y	-1.2111- 2.581i	-0.368 + 0.236i	0.568 + 0.007i
4x	0.569 + 0.163i	0.637 + 0.066i	0.734 + 0.057i
5x	0.554 + 0.220i	0.668 + 0.054i	-0.823 - 0.056i
5y	-0.987 - 1.562i	-0.243 + 0.123i	0.455 + 0.049i
7x	0.272 + 0.056i	0.361 + 0.026i	0.479 + 0.035i
8x	0.211 + 0.104i	0.384 + 0.014i	-0.534 - 0.023i
8y	-0.617 - 0.882i	-0.128 + 0.081i	0.292 + 0.023i
10x	0.152 + 0.019i	0.165 - 0.005i	0.312 - 0.004i
11x	0.149 + 0.024i	0.162 - 0.001i	-0.324 + 0.001i
11y	-0.206 - 0.409i	-0.053 + 0.035i	0.189 + 0.003i
13x	0.030 - 0.006i	0.049 - 0.005i	0.145 - 0.002i
14x	0.021 - 0.001i	0.0498- 0.005i	-0.150 + 0.003i
14y	-0.064 - 0.107i	-0.016 + 0.012i	0.088 + 0.001i
19x	1.996 + 0.770i	1.387 + 0.286i	-0.066 - 0.005i
19y	-1.838 - 3.988i	-0.524 + 0.182i	0.261 + 0.012i

Table 3. First three modes (natural frequencies, damping ratios and modal vectors) estimated using the state space model 2 and the EM algorithm.

		State space model 1		
		mode 1	mode 2	mode 3
State space model 2	mode 1	0,9110	0,2443	0,1031
	mode 2	0,0656	0,9847	0,0191
	mode 3	0,1019	0,0286	0,9994

Table 4. MAC values between modal vectors computed using the state space model 1 and state space model 2 (for DOFs 13x, 14x and 14y in model 1, results of setup 2 has been used).

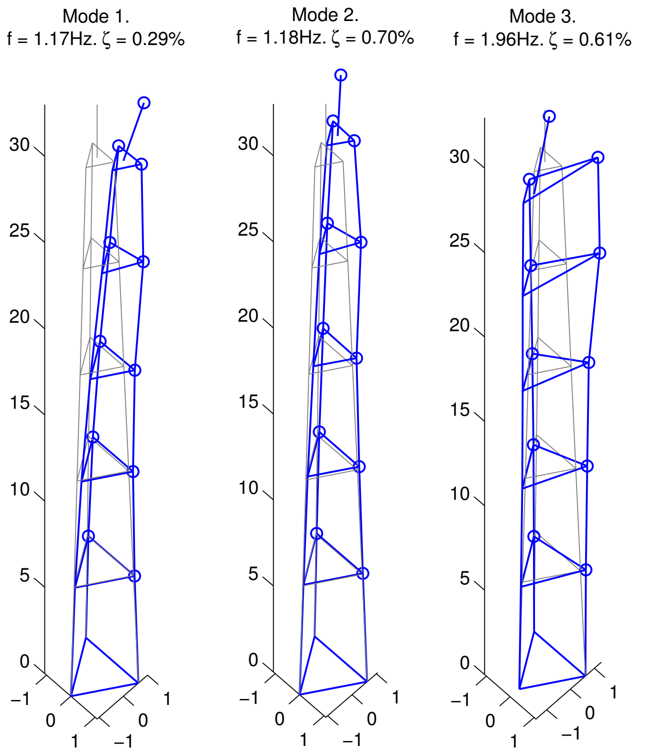

Figure 3. Plot of the modal vectors (real part) corresponding to the first three modes estimated using the state space model 2 and the EM algorithm.

Both models, state space model 1 and 2, were estimated using the EM algorithm detailed in Section 2.3. The state space order used for both models was $n_s = 16$ (in theory, 8 modes were identified). The first three modes are shown in Tables 2 and 3. Figure 3 shows the first three modes estimated using the state space model 2. The following points can be highlighted:

• The global modal vectors are obtained directly with both models, no post-process is needed. Besides, only one value for the natural frequency and damping ratio per mode is obtained.

• The global modal vectors estimated with both models are similar. The modal vectors are compared in Table 4 by mean of the MAC value.

• It is important to note that DOFs 13x, 14x and 14y (13x means node 13, X-direction, and so on) were measured in setups 2 and 3, although they were not permanent sensors. If we use the state space model 1, the modal vectors are estimated in these DOFs in both setups (this is indicated in Table 2 by mean of a superscript). However, if we use the state space model 2, only one value for 13x, 14x and 14y is obtained. And what is more important, this model use the data of setup 2 and setup 3 to estimate the most likelihood modal vectors at these DOFs.

4 CONCLUSIONS

The estimation of the modal parameters from data recorded at different setups of sensors it is not trivial task. One option is to process each setup separately so the modal vectors are estimated *by parts*. Then, the global modal vectors are obtained gluing these parts.

A more interesting option is to use an adequate state space model to process all the setups at the same time. The result is that the global modal vectors are obtained directly. In this work we have analysed and compared two state space models that can be used with these multiple data.

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